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Equilibrium—Its Connection to Global Integrability Conditions for Stationary Einstein–Maxwell Fields

J. P. WARD

Mathematics Department, Cranfield Institute of Technology, Cranfield, Bedford England*

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Abstract

This paper describes the intimate connection between global integrability conditions and equilibrium conditions for stationary Einstein-Maxwell fields. Regularity conditions are deduced for all the presently known classes of solutions.

1. *Introduction*

Both Majumdar (1947) and Levy (1968) have considered integrability conditions in Einstein-Maxwell theory; Levy restricted himself to stationary, axially symmetric vacuum solutions[1] while Majumdar only considered static, electrovacuum solutions. In this paper I extend these considerations to stationary Einstein-Maxwell solutions. We shall see that the local integrability conditions are satisfied automatically (in the exterior), but the global integrability conditions are not and in general specify two independent constraints on the parameters describing the solutions. A physical interpretation of these integrability conditions is attempted and we show that it is plausible to regard these conditions as equilibrium conditions that have to be satisfied by the sources of the relativistic field. This interpretation is aided by comparison with Newtonian theory and by use of a spinning test particle analysis.

In section 2 we introduce the field equations used in this work. Here also, necessary conditions for regularity of solutions to these equations are given. An interpretation of one of the two conditions that arise is given. In section 3 we briefly indicate how the various classes of exact solutions arise in the present formalism. We then determine, in section 4, what constraints, if any, are to be placed on the parameters describing the exact solutions in order that the conditions of integrability be satisfied. In section 5 we use a spinning test

* Formerly Queen Elizabeth College, University of London.

[1] Levy only considers integrability of ν; cf. equation (3.3).

particle analysis to indicate that the remaining integrability conditions can, in general, also be regarded as an equilibrium condition. The paper ends with a conclusion.

2. *Einstein-Maxwell Field Equation*

I will use the notation of Israel and Wilson (1972). The metric of a general stationary field can be expressed in the form

$$ds^2 = -f^{-1}\gamma_{mn}\, dx^m\, dx^n + f(dx^4 + \omega_m\, dx^m)^2 \tag{2.1}$$

then Einstein-Maxwell field equations,

$$R_{\alpha\beta} = -8\pi E_{\alpha\beta}, \qquad 4\pi E_{\alpha\beta} = -F_\alpha{}^\epsilon F_{\beta\epsilon} + \tfrac{1}{4} g_{\alpha\beta} F_{\epsilon\mu} F^{\epsilon\mu} \tag{2.2}$$

$$F_{[\alpha\beta;\epsilon]} = 0, \qquad F^{\alpha\beta}_{,\beta} = J^\alpha = 0 \tag{2.3}$$

reduce to[2]

$$f\nabla^2 E = \nabla E \,.\, (\nabla E + 2\Psi^*\nabla\Psi) \tag{2.4}$$

$$f\nabla^2 \Psi = \nabla\Psi \,.\, (\nabla E + 2\Psi^*\nabla\Psi) \tag{2.5}$$

$$-f^2 R_{mn}(\gamma) = \tfrac{1}{4} E_{(,m} E^*_{,n)} + \Psi E_{(,m} \Psi^*_{,n)} - E\Psi_{(,m}\Psi^*_{,n)} + \text{c.c.} \tag{2.6}$$

$$f = \tfrac{1}{2}(E + E^*) + \Psi\Psi^* \tag{2.7}$$

where

$$E = f - \Psi\Psi^* + i\phi \quad \text{and} \quad \Psi = A_4 + i\Phi \tag{2.8}$$

The scalar function ϕ is determined from the equation

$$-f^2 \nabla \wedge \boldsymbol{\omega} = i(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) + \nabla\phi \tag{2.9}$$

$R_{mn}(\gamma)$ is the Ricci tensor formed from the positive definite three-dimensional metric γ_{mn}. A_4 and Φ are electromagnetic potentials and they define the electromagnetic field tensor through the expression

$$\begin{aligned} F_{\alpha\beta} + iF^*_{\alpha\beta} = {} & [i\epsilon_{abc}\delta_\alpha{}^a \delta_\beta{}^b \gamma^{1/2} f^{-1} + (\delta_\alpha{}^4 \delta_\beta{}^p - \delta_\alpha{}^p \delta_\beta{}^4) \\ & + \omega_e(\delta_\alpha{}^e \delta_\beta{}^p - \delta_\beta{}^e \delta_\alpha{}^p)]\, \Psi^*_{,p} \end{aligned} \tag{2.10}$$

Eliminating ϕ from (2.8) we have

$$\nabla \wedge \boldsymbol{\omega} = -if^{-2}(\tfrac{1}{2}\nabla E^* + \Psi\nabla\Psi^*) + \text{c.c.} \equiv \mathbf{W} \tag{2.11}$$

[2] Greek indices run from 1 to 4 and Latin from 1 to 3. ∇ is the usual gradient operator, ∇^2 the Laplacian using γ_{mn} as base metric. A comma denotes partial differentiation, a semicolon and a stroke $|$ denotes covariant differentiation with respect to $g_{\alpha\beta}$ and γ_{mn}, respectively. An asterisk * on a scalar means complex conjugate, while if it appears on a tensor it refers to the dual object. ϵ_{abc} is the three-dimensional, completely skew-symmetric object with $\epsilon_{123} = 1$.

The three-dimensional Einstein tensor constructed from γ_{mn} is easily deduced from $R_{mn}(\gamma)$ and we find

$$G_{mn}(\gamma) = -\tfrac{1}{4}f^{-2}\,[E_{,m}E^*_{,n} + 2\Psi^*(E^*_{,n}\Psi_{,m} + E^*_{,m}\Psi_{,n}) - 2E(\Psi_{,m}\Psi^*_{,n} + \Psi^*_{,n}\Psi_{,m}) - \tfrac{1}{2}\gamma_{mn}(E_{,p}E^{*,p} + 4\Psi^*E^{*,p}\Psi_{,p} - 4E\Psi^{,p}\Psi^*_{,p})] + \text{c.c} \equiv H_{mn} \tag{2.12}$$

So given solutions to equations (2.4) and (2.5) we can find solutions (locally) to equations (2.6) and (2.11) for γ_{mn} and $\boldsymbol{\omega}$, respectively, if the following integrability conditions are satisfied by the vector **W** and the set of components H_{mn}:

$$\nabla . \mathbf{W} = 0 \tag{2.13}$$

and

$$H^{mn}|n = 0 \tag{2.14}$$

Making the calculation implied in equations (2.13) and (2.14) we find

$$\nabla . \mathbf{W} = if^{-3}\{\tfrac{1}{2}[f\nabla^2 E - \nabla E . (\nabla E + 2\Psi^*\nabla\Psi)] + \Psi^*[f\nabla^2\Psi - \nabla\Psi . (\nabla E + 2\Psi^*\nabla\Psi)]\} + \text{c.c.} \tag{2.15}$$

and

$$H^{ip}_{|i} = -f^{-3}\{(\tfrac{1}{4}E^{*,p} + \tfrac{1}{2}\Psi\Psi^{*,p})[f\nabla^2 E - \nabla E . (\nabla E + 2\Psi^*\nabla\Psi)] + \tfrac{1}{2}[\Psi^*E^{*,p} - \Psi^{*,p}(E + E^*)][f\nabla^2\Psi - \nabla\Psi . (\nabla E + 2\Psi^*\nabla\Psi)]\} + \text{c.c.} \tag{2.16}$$

Thus the local integrability conditions are satisfied automatically because of equations (2.4) and (2.5). However, as pointed out by Szekeres (1968) (there for the static, vacuum, axisymmetric case), the global integrability conditions (G.I.C.) ensuring the existence of $\boldsymbol{\omega}$ and γ_{mn} over some finite domain, are more severe. The global condition arising from (2.11) is simply expressed as

$$\int_S \mathbf{W} . d\mathbf{S} = 0 \tag{2.17}$$

where S is any closed two-surface in the exterior. [The exterior is here defined as those points of space for which equations (2.4) and (2.5) are satisfied.] The validity of (2.17) assumes only that $\boldsymbol{\omega}$ is C^1 on S and follows from an application of Stokes' theorem to an exterior closed two-surface S. The G.I.C. arising from (2.12) is that

$$\int_S H^{ij}\lambda_i n_j\, dS = 0 \tag{2.18}$$

where λ_i is a vector satisfying $G^{ij}\lambda_{i|j} = 0$ and S is any closed two-surface in the exterior. For the static vacuum case this condition was given by Muller Zum

Hagen (1974). Condition (2.18) follows by integrating $\lambda_j(G^{ij} - H^{ij})$ over the surface S and then using the divergence theorem on $\int_S G^{ij}\lambda_j n_i \, dS$ and using the identity $G^{ij}_{|j} \equiv 0$.

A possible interpretation of the physical meaning of (2.18) can be gleaned from Newtonian theory. Thus, following Bondi and Morgan (1970), consider Newtonian theory with potential ϕ, satisfying $\overline{\nabla}^2\phi = 0$, where $\overline{\nabla}^2$ is the flat-space Laplacian operator. Bondi has constructed a stress tensor for this theory:

$$T_{ij} = (1/4\pi G)[\phi_{,i}\phi_{,j} - \tfrac{1}{2}\delta_{ij}(\phi_{,m})^2] \tag{2.19}$$

The divergence of this tensor is minus the gravitational force acting at a point, and the total force acting over a surface S is therefore

$$F^i = \int_S T^{ij} n_j \, dS \tag{2.20}$$

In a static system this force is required to vanish when evaluated over any surface. For the purposes of generalization to a nonflat metric (where the integrands should be invariants) transvect (2.20) with a constant vector $v_i(v_{i,j} = 0)$; then (2.20) becomes

$$F^i v_i = \int_S T^{ij} v_i n_j \, dS \tag{2.21}$$

So conditions of equilibrium valid in Newtonian theory are that

$$\int_S T^{ij} v_i n_j \, dS = 0 \tag{2.22}$$

What I wish to do now is to seek for a generalization of the above stress tensor, (2.19), to include relativistic effects. It is fairly clear that static vacuum fields of Einstein's theory correspond most closely to Newtonian situations. The field equations describing a general static field in Einstein's theory are obtainable from equations (2.4)-(2.7). We find they are

$$E\nabla^2 E - (\nabla E)^2 = 0 \tag{2.23}$$

or equivalently

$$\nabla^2(\log E) = 0 \tag{2.24}$$

and

$$R_{mn}(\gamma) = -\tfrac{1}{2}(\log E)_{(,m}(\log E)_{,n)} \tag{2.25}$$

where for the static case $f = g_{00} = E$. The Einstein tensor constructed from equation (2.25) is

$$G_{mn}(\gamma) = -\tfrac{1}{2}[(\log E)_{,m}(\log E)_{,n} - \tfrac{1}{2}\gamma_{mn}(\log E)_{,p}(\log E)^{,p}] \tag{2.26}$$

Now for weak fields $(\log E)$ is a very good approximation to the Newtonian potential ϕ (up to a multiplicative constant). It is now clear that $G_{ij}(\gamma)$ is the natural generalization of the stress tensor T_{ij}. We can exploit this correspondence for strong fields and for systems that include Maxwell fields as well. The meaning of the G.I.C. (2.18) is now clear. It represents a necessary equilibrium condition that the sources of the relativistic field must satisfy. Some support for this interpretation will be given in later sections.

It is not clear what physical interpretation to put on the global condition given in (2.17). It seems to have no Newtonian analog. However, in section 5, a spinning test particle analysis is employed to show that this condition can also be regarded as an equilibrium condition and refers mainly to the equilibrium of spinning sources.

3. *Application to Exact Solutions*

It will be of some use to describe briefly how to generate the presently known classes of solutions using the present formalism.

All the known classes of solutions to the stationary Einstein-Maxwell equations reduce to solving one (or two) Laplace equations. All follow straightforwardly if we assume E to be an analytic function of Ψ.[3] It then follows from equations (2.4) and (2.5) that, independently of any assumption of spatial symmetry,

$$E = 1 - \Psi/a$$

where a is a complex constant.[4] Then both equations (2.4) and (2.5) reduce to

$$f\nabla^2 E = (\nabla E)^2(1 - 2aa^* + 2aa^*E^*) \tag{3.1}$$

while (2.6) becomes

$$-f^2 R_{mn}(\gamma) = \tfrac{1}{2}(1 - 4aa^*)E_{(,m}E^*_{,n)} \tag{3.2}$$

These equations are still enormously difficult to solve and further simplifications are necessary. In the present context we shall consider only two such simplifications.

(A) Reduction to Axial Symmetry. For this case it is well known that the three-dimensional metric γ_{ab} can be taken in the form [this requires the use of the field equation $R_3{}^3 + R_4{}^4 = -\gamma^{33} f R_{33}(\gamma)$]

$$\gamma_{ab} = \mathrm{diag}\,(e^{2\nu}, e^{2\nu}, r^2)$$

where we choose coordinates

$$(z, r, \phi, t) \rightarrow (x^1, x^2, x^3, x^4)$$

We can also assume without loss of generality that

$$\omega_i = (0, 0, \omega)$$

[3] First considered by Ernst (1968), for the axisymmetric case.

[4] Here we fix one of the two arbitrary complex constants that arise here by demanding $E \rightarrow 1$ as $\Psi \rightarrow 0$.

With this simplification to axial symmetry, the form of (3.1) remains the same, but now, of course, the operators are written with respect to the new γ_{ab}. Also equation (3.2) is much simplified and reduces to the following equation which allows for the determination of ν by simple quadrature:

$$r(\nabla.\boldsymbol{\nu})_m = \delta_m{}^1(\nu_{,2}) + \delta_m{}^2(-\nu_{,1}) \equiv \mathbf{V} \tag{3.3}$$

where $\boldsymbol{\nu} = (0, 0, \nu)$ and

$$8\nu_{,1} = rf^{-2}[(E_{,1}E^*_{,2} + E_{,2}E^*_{,1}) + 4\Psi(E_{,1}\Psi^*_{,2} + E_{,2}\Psi^*_{,1}) - 4E(\Psi_{,1}\Psi^*_{,2} + \Psi_{,2}\Psi^*_{,1})] + \text{c.c.} \tag{3.4}$$

$$8\nu_{,2} = -rf^{-2}[(E_{,1}E^*_{,1} - E_{,2}E^*_{,2}) + 4\Psi(E_{,1}\Psi^*_{,1} - E_{,2}\Psi^*_{,2}) - 4E(\Psi_{,1}\Psi^*_{,1} - \Psi_{,2}\Psi^*_{,2})] + \text{c.c.} \tag{3.5}$$

Similarly the function ω can now be evaluated from equation (2.11) once a solution to equation (3.1) is given.

(B) Reduction to a Flat Background. The metric γ_{ab} will be flat iff $R_{mn}(\gamma) = 0$. [This implies $R_{mnpq}(\gamma) = 0$.] This restriction is in some ways a degenerate case of axial symmetry since $\nu \equiv 0$. However, our differential operator ∇ is now more general since we now allow differentiations with respect to all three spatial coordinates. We note that this case is generated by taking $4aa^* = 1$.

With one or another of the above restrictions (A) or (B), all of the known classes of solutions can now be derived.

To obtain the Weyl vacuum, Weyl electromagnetic, Majumdar-Papapetrou, and P.I.W.[5] classes of solutions, we transform equation (3.1) by introducing a new complex function χ by

$$\nabla\chi = \frac{\nabla E}{E(1 - 2aa^*) + aa^*(1 + E^2)} \equiv \mathbf{P} \tag{3.6}$$

The existence of χ is guaranteed (locally) since $\nabla.\mathbf{P} = 0$. [We also note that the denominator in equation (3.6) is the functional form of f with E^* replaced by E.] With this transformation, (3.1) becomes

$$2[E(1 - 2aa^*) + aa^*(1 + E^2)]^2 f\nabla^2\chi = (\nabla E)^2(E - E^*)(1 - 4aa^*) \tag{3.7}$$

It is now clear from this equation that a harmonic function will generate solutions of the field equations for the following cases:

(a) $E = E^*$: This condition gives rise to the Weyl, Majumdar-Papapetrou static solutions. If we define $k = 1 - 4aa^*$, then there are three particular

[5] This class recently discovered by Perjes (1971) and independently by Israel and Wilson (1972).

cases to consider:

(1) $k = 1$: Weyl vacuum class (Bonnor, 1953)

(2) $k = 0$: Majumdar-Papapetrou class (Papapetrou, 1947)
$R_{mnps}(\gamma) = 0, \quad |e_i| = M_i$

(3) $k \neq 0, 1$: Weyl electromagnetic class
$|e_i| = hM_i, \quad h = \text{const} \neq 1$

In the above e_i and M_i are parameters corresponding, respectively, to charge and mass of individual sources of the system. In (2) and (3) it is assumed that all the charges have the same sign.

(b) $k = 0$. This condition gives rise to the P.I.W. class of solutions. The characteristics of this class are that

$$R_{mnps}(\gamma) = 0, \qquad |e_i| = M_i$$

and for every particle in the system $\mathbf{h}_i = +\boldsymbol{\mu}_i$ or $\mathbf{h}_i = -\boldsymbol{\mu}_i$, where $\mathbf{h}$ and $\boldsymbol{\mu}$ correspond, respectively, to magnetic dipole moment and angular momentum.

This is as far as one can go with the transformation (3.6). However, there are two more classes of interest; the Papapetrou rotating solutions of Papapetrou (1947) and the class recently discovered by Bonnor (1973).

To generate the Papapetrou class, we return to equation (3.1) and define a new complex potential ξ by

$$E = \frac{k^{1/2}\xi - 1}{k^{1/2}\xi + 1}, \qquad k \neq 0 \tag{3.8}$$

Then equation (3.1) becomes

$$2(\nabla\xi)^2\xi^* - (\xi\xi^* - 1)\nabla^2\xi = 0 \tag{3.9}$$

The electromagnetic generalization (vacuum solutions given by choosing $k = 1$) of the Papapetrou rotating solution is obtained by taking $\xi = -i \coth \Delta$ ($\Delta = \Delta^*$) for which equation (3.9) gives

$$\nabla^2\Delta = 0 \tag{3.10}$$

The physical characteristics of this class are well known. The mass monopole term must be taken as zero in order that the metric be Minkowskian at infinity. The NUT solution is in this class, generated by the monopole solution of the above Laplace equation.

The class recently discovered by Bonnor seems very difficult to derive using the present formalism. However, in the present context, only his results are of interest. He finds that the whole of his solution is generated by one harmonic function B, $\nabla^2 B = 0$, where

$$B = \int \frac{dv}{f} \qquad \text{and} \qquad f = g_{00} = \alpha^{-2}(2 + g \sin v + h \cos v)$$

Here α, g, h are arbitrary constants. These solutions are axially symmetric and seem to resemble the Papapetrou solutions very closely.

4. *Regularity Conditions for Exact Solutions*

We now address ourselves to the investigation of the constraints placed on the parameters describing the exact solutions as a result of satisfying the G.I.C. given in equations (2.17) and (2.18). Throughout this section we are only interested in axially symmetric fields (or those for which $R_{mn}(\gamma) = 0$) for which $E = 1 - \Psi/a$. Equations (2.17) and (2.18) then reduce to, respectively,

$$\int_S \left[-\frac{i}{4} f^{-2} \nabla E^*(1 + E) + \text{c.c.} \right] . \, d\mathbf{S} = 0 \tag{4.1}$$

and

$$\int_S f^{-2}(1 - 4aa^*)[2E^{(,m}E^{*,n)} - \gamma^{mn}E_{,p}E^{*,p}] \lambda_m n_n \, d\mathbf{S} = 0 \tag{4.2}$$

where in (4.2) $G^{mn}\lambda_{m|n} = 0$.

For axially symmetric situations a vector λ_m satisfying $G^{mn}\lambda_{m|n} = 0$ is $\lambda^m = (1, 0, 0)$. This was given by Muller Zum Hagen for the static vacuum case but this vector suffices for this more general case also (cf. Ward, 1974). So for the axisymmetric case (4.2) can be written

$$\int_S -\tfrac{1}{4} f^{-2}(1 - 4aa^*)[2E^{(,1}E^{*,n)} - \delta^{1n}E_{,p}E^{*,p}] n_n \, dS = 0 \tag{4.2a}$$

When these conditions are applied to the classes of solutions described earlier, we arrive at the following results:

A(a). Weyl Vacuum class. For this class $E = E^*$ and $k \neq 0$. These restrictions imply that condition (4.1) is vacuous, while (4.2a) becomes [note equation (3.6)]

$$\int_S -\frac{k}{2} [\chi^{,1}\chi^{,n} - \tfrac{1}{2}\delta^{1n}\chi_{,p}\chi^{,p}] n_n \, dS = 0 \tag{4.3}$$

and since $\nabla^2\chi = 0$ then (4.3) is easily interpreted. Regarding χ as a Newtonian potential condition (4.3) states that the component of force in the z direction on the surface S is required to vanish.

A(b). Majumdar–Papapetrou class. Here $E = E^*$ and $k = 0$. Thus both conditions (4.1) and (4.2) are automatically satisfied. Systems of particles described by this class are always in equilibrium. This is to be expected from Newtonian considerations because of the special relation between the mass and charge parameters in these solutions.

A(c). Weyl electromagnetic class. Here $E = E^*$ and $k = 1$. We arrive at the same conclusions as for A(a). This is also to be expected, since in Newtonian theory the force laws of electromagnetism and gravitation are the same (up to sign).

For this class of solutions, all the charges e_i have the same sign and $|e_i| = dm_i$; $d = \text{const}$.

(B). The P.I.W. solutions. Here $k = 0$ and thus condition (4.2) is automatically satisfied, while (4.1) reduces to (note $\nabla^2\chi = 0$)

$$\int_S \frac{i}{4}(\chi\nabla\chi^* - \chi^*\nabla\chi) \,.\, d\mathbf{S} = 0 \tag{4.4}$$

We shall attempt an interpretation of this condition in section 5.

(C). The Papapetrou rotating solutions. In terms of the potential Δ, where $\nabla^2\Delta = 0$, conditions (4.1) and (4.2a) reduce to ($k \neq 0$)

$$\int_S \frac{8}{k^{1/2}}(\nabla\Delta) \,.\, d\mathbf{S} = 0 \tag{4.5}$$

and

$$\int_S -\tfrac{1}{2}[\Delta^{,1}\Delta^{,n} - \tfrac{1}{2}\delta^{1n}(\Delta_{,n})(\Delta^{,n})]\, n_n \, dS = 0 \tag{4.6}$$

The interpretation of (4.6) is the same as given in A(a). Condition (4.5) is an interesting restriction and implies that the monopole contribution to Δ is to be taken as zero. This is of course consistent with the usual restrictions placed on the Papapetrou class of solutions. (This restriction essentially eliminates NUT-type solutions.)

(D). The Bonnor class. For this class (4.1) and (4.2a) reduce to (again $\nabla^2 B = 0$)

$$\int_S \nabla B \,.\, d\mathbf{S} = 0 \tag{4.7}$$

and

$$(g^2 + h^2 - 4)\int_S (B^{,1}B^{,n} - \tfrac{1}{2}\delta^{1n}B_{,p}B^{,p})n_n \, dS = 0 \tag{4.8}$$

As Bonnor points out, restricting g and h so that $g^2 + h^2 = 4$ takes his class into a subset of the P.I.W. class. The restrictions placed on B are the same as in the Papapetrou class.

5. *Spinning Test Particles in P.I.W. Fields*

We now come to the interpretation of the condition given in equation (2.17) for the general case and in equations (4.4), (4.5), and (4.7) for particular classes of solutions. For reasons of brevity I shall only discuss the conditions arising in equation (4.4) since the P.I.W. class of solutions has aroused considerable interest recently. There are two main solutions of this class at present.

The first, due to Bonnor and Ward (1972),[6] describes the field of two "Perjeons." This solution is generated by the choice of harmonic potential

$$\chi = 1 + \frac{m_1}{r_1} + \frac{m_2}{r_2} + i\left[\frac{\mu_1(z+a)}{r_1{}^3} + \frac{\mu_2(z-a)}{r_2{}^3}\right] \tag{5.1}$$

where

$$r_1{}^2 = x^2 + y^2 + (x+a)^2$$
$$r_2{}^2 = x^2 + y^2 + (z-a)^2$$

The solution for $\boldsymbol{\omega}$ is of main interest and they find (written in Cartesian coordinates)

$$\begin{aligned}\boldsymbol{\omega} = (y\hat{i} - x\hat{j})\Bigg\{&\frac{m_1\mu_1}{r_1{}^4} + \frac{m_2\mu_2}{r_2{}^4} + \frac{2\mu_1}{r_1{}^3} + \frac{2\mu_2}{r_2{}^3} + \frac{m_2\mu_1}{r^2}\left[\frac{r_2}{r_1}\left(\frac{1}{2a^2}\right)\right.\\ &\left.+ \frac{(z+a)}{ar_1{}^3r_2}(r^2 + z^2 - a^2)\right] + \frac{m_1\mu_2}{r^2}\left[\frac{r_1}{r_2}\left(\frac{1}{2a^2}\right) - \frac{(z-a)}{ar_2{}^3r_1}\right.\\ &\left.\times (r^2 + z^2 - a^2)\right]\Bigg\}\end{aligned} \tag{5.2}$$

where $r^2 = x^2 + y^2$.

This solution corresponds to the field of two massive, spinning, charged, magnetic particles. At $r = 0$, $z = a$, there is a particle of mass m_2 (numerically equal to the charge) and angular momentum μ_2 (numerically equal to magnetic moment) while at $r = 0$, $z = -a$, there is a particle of mass m_1 and angular momentum μ_1.

The second solution is due to Spanos (1974), who constructs the field due to two "$e^2 = M^2$" Kerr-Newman sources generated by the harmonic potential

$$\chi = 1 + \sum_{A=1}^{2} \frac{M_A}{R_A} \tag{5.3}$$

where

$$R_A{}^2 = r^2 + z_A{}^2$$
$$z_A = z + (-1)^A b + ia_A$$

The solution he finds is (in axisymmetric coordinates)

$$\begin{aligned}\omega_\phi = -2g_m \sum_{A=1}^{2} \Bigg[&\frac{M_A Z_A}{R_A} + \frac{M_A{}^2 Z_A}{Z_A^{*2} - Z_A{}^2}\left(\frac{R_A^*}{R_A} - 1\right)\\ &+ \frac{M_A M_{3-A} Z_{3-A}}{(Z_A^{*2} - Z_{3-A}^2)}\left(\frac{R_A^*}{R_{3-A}} - 1\right)\Bigg]\end{aligned} \tag{5.4}$$

[6] This solution has been considerably generalized by Ward (1973).

where $\boldsymbol{\omega} = (0, 0, \omega_\phi)$ in (z, r, ϕ) coordinates. There is a particle (actually source has the geometry of a ring in the background coordinates) in the plane $z = b$ of mass M_1 and angular momentum $M_1 a_1$. Similarly in the plane $z = -b$, there is a particle of mass M_2 and angular momentum $M_2 a_2$.

In this context the main interest in these solutions is the presence of line singularities on various portions of the axis of symmetry.

Bonnor and Ward find that [here written in axisymmetric coordinates (z, r, ϕ)]

$$\omega_\phi(0, z) = \begin{cases} 0 & |z| > a \\ \dfrac{m_2\mu_1 + m_1\mu_2}{2a^2} & |z| < a \end{cases} \tag{5.5}$$

Similarly Spanos' solution implies[7]

$$\omega_\phi(0, z) = \begin{cases} 0 & |z| > b \\ \dfrac{4M_1M_2(a_1 + a_2)}{4b^2 + (a_1 + a_2)^2} & |z| < b \end{cases} \tag{5.6}$$

Since regularity on the axis demands that $\omega_\phi(0, z) = 0$, both of these solutions have line singularities. For regularity of the exterior geometry of their solutions they require that, in both solutions, the total angular momentum per unit mass should be zero; that is,

$$m_2\mu_1 + m_1\mu_2 = 0 \qquad \text{for Bonnor, Ward} \tag{5.7}$$

and

$$M_1M_2(a_1 + a_2) = 0 \qquad \text{for Spanos} \tag{5.8}$$

If one uses the G.I.C. (4.4) in conjunction with equations (5.1) and (5.3) then one obtains, as expected, precisely the conditions given in (5.7) and (5.8). Attempting to interpret the meaning of the restrictions given in equation (5.7), Bonnor and Ward used a test particle analysis and found that when such a particle, endowed with charge and mass, was placed in the field of a single Perjeon [put $m_2 = \mu_2 = 0$ in (5.1) and (5.2)] it was acted on by a force that did not vanish unless the angular momentum, μ_1, of the source field vanished. Although their work gave some credence in the interpretation of (5.7) as an equilibrium condition, it was by no means conclusive. It is my purpose in the present section to generalize their work by considering the action of a single Perjeon on a test particle endowed with mass, charge, angular momentum, and magnetic dipole moment. To treat such a test particle, we require the use of the Papapetrou (1951) equations of motion (amended slightly to encom-

[7] This corrects the results reached by Spanos.

pass magnetic dipole effects). The Papapetrou equations are

$$\frac{D}{DS}\left(mu^{\mu}+u_{\rho}\frac{D}{DS}S^{\mu\rho}\right)=-\tfrac{1}{2}R^{\mu}{}_{\omega\rho\sigma}u^{\nu}S^{\rho\sigma}-F^{\mu\nu}J_{\nu}+F^{*\mu\beta}{}_{;\gamma}p^{\gamma}u_{\beta} \tag{5.9}$$

$$\frac{D}{DS}S^{\mu\nu}+u^{\mu}u_{\rho}\frac{D}{DS}S^{\nu\rho}-u^{\nu}u_{\rho}\frac{D}{DS}S^{\mu\rho}=2F^{*\gamma[\mu}p^{\nu]}u_{\gamma} \tag{5.10}$$

with Matthison condition (cf. also Pirani, 1956)

$$S^{\mu\rho}u_{\nu}=0 \tag{5.11}$$

where $D/DS = ;\mu u^{\mu}$ and $u^{\mu}=dx^{\mu}/ds$ is the test-particles four-velocity. $S^{\mu\nu}$ is the skew-symmetric spin tensor of the particle and because of (5.11) has only three independent components, which are associated with the angular momentum of the test particle. Equation (5.9) is just the generalization of Newton's equation of motion, the L.H.S. being the rate of change of total momentum (including that due to its spin). The R.H.S. are the applied forces, $F^{\mu\nu}J_{\nu}$ being the usual Lorentz force and $F^{*\mu\beta}_{;\gamma}p^{\gamma}u_{\beta}$ the covariant generalization of $(\mathbf{p}\cdot\nabla)\mathbf{B}$ which is the classical force on a magnetic dipole $\mathbf{p}$ in a field $\mathbf{B}$.[8] The L.H.S. of (5.10) is essentially the rate of change of angular momentum and is to be equated with the applied torque, $2F^{*\gamma[\mu}p^{\nu]}u_{\gamma}$, which is just the covariant generalization of $\mathbf{p}\wedge\mathbf{B}$.

The whole of this section then, will be concerned with the evaluation of the force on a test particle mainly using (5.9). More precisely we shall consider the effect on a spinning test particle when placed on the axis of symmetry in the field of a single Perjeon given by taking $m_2=\mu_2=0$ in (5.1) and (5.2). Thus the metric we use is

$$ds^2=-f^{-1}(dz^2+dr^2+r^2d\phi^2)+f(dt+\omega_{\phi}\,d\phi)^2$$

where

$$f^{-1}=\left(1+\frac{m_1}{r_1}\right)^2+\left[\frac{\mu_1(z+a)}{{r_1}^3}\right]^2$$

$$\omega_{\phi}=-r^2\left(\frac{m_1\mu_1}{{r_1}^4}+\frac{2\mu_1}{{r_1}^3}\right)$$

$$r_1=+\left[x^2+y^2+(z+a)^2\right]^{1/2} \tag{5.12}$$

The spin vector S^{α} and magnetic moment vector p^{α} of the test particle are taken to be

$$S^{\alpha}=(S^1,0,0),\qquad p^{\alpha}=(p^1,0,0)$$

[8] First suggested to me by Dr. D. Rawson-Harris.

and are thus aligned along the axis of symmetry. The spin vector S^α is related to $S^{\rho\sigma}$ by (cf. Pirani, 1956)

$$S^\mu = -\tfrac{1}{2}\eta^{\mu\nu\rho\sigma}u_\nu S_{\rho\sigma} \tag{5.13}$$

or

$$S^{\mu\nu} = -\eta^{\mu\nu\rho\sigma}S_\rho u_\sigma$$

The physical components of S^μ for an observer moving with the test particle are the components of angular momentum of the test body (this is easily seen from the definition of $S^{\mu\nu}$ given in Papapetrou's original paper). Thus the physical components of S^α and p^α are $S^{(A)}$ and $p^{(A)}$, respectively, where

$$S^{(A)} = \lambda_\alpha^{(A)}p^\alpha \qquad \text{and} \qquad p^{(A)} = \lambda_\alpha^{(A)}p^\alpha$$

where $\lambda_\alpha^{(A)}$ is an orthonormal tetrad moving with the test particle.

$$\lambda_{(1)}^\alpha = (f^{1/2}, 0, 0, 0), \qquad \lambda_{(2)}^\alpha = (0, f^{1/2}, 0, 0)$$
$$\lambda_{(3)}^\alpha = (0, 0, a, b), \qquad \lambda_{(4)}^\alpha = (0, 0, 0, f^{-1/2})$$

where a and b are to be determined from the orthogonality condition $\lambda_{(A)}^\alpha\lambda_\beta^{(A)} = \delta_\beta{}^\alpha$. Because of our choice of S^α, it follows that the only non-vanishing components of $S^{\alpha\beta}$ are S^{24} and S^{23}, where

$$S^{24} = \left(\frac{f^{3/2}\omega}{r}\right)S_1, \qquad S^{23} = -\left(\frac{f^{3/2}}{r}\right)S_1 \tag{5.14}$$

It is now easy to show (Ward, 1974), with the aid of equations (5.10) and (5.11) and employing the particular characteristics of the metric we are using, that equation (5.9) can be put in the form

$$\frac{D}{DS}(mu^\mu) - S^{\mu\nu}\frac{D^2u_\nu}{DS^2} + \tfrac{1}{2}R^\mu{}_{\nu\rho\sigma}u^\nu S^{\rho\sigma} = -F^{\mu\nu}J_\nu + F^{*\mu\beta}{}_{;\gamma}p^\gamma u_\beta \tag{5.15}$$

where both sides of this equation are to be evaluated at $r = 0$, $z = a$; the position of the test particle. The important member of (5.15) is $\mu = 1$ (we are only interested in the initial force in the z-direction). We see that the second term in (5.15) drops out leaving

$$\frac{d}{ds}\left(m\frac{dz}{ds}\right)_{z=a} + [m\Gamma_{44}^1(u^4)^2 + u^4S^{\rho\sigma}(\Gamma_{\sigma4,\rho}^1 + \Gamma_{\rho\alpha}^1\Gamma_{\sigma4}^\alpha)]_{z=a} = [-F^{1\alpha}J_\alpha$$
$$+ F^{*\mu\beta}{}_{;\gamma}p^\gamma u_\beta\delta_\mu{}^1]_{z=a} \tag{5.16}$$

because

$$\tfrac{1}{2}R^\mu{}_{\nu\rho\sigma}u^\nu S^{\rho\sigma} = u^\nu S^{\rho\sigma}(\Gamma_{\sigma\nu,\rho}^\mu + \Gamma_{\rho\alpha}^\mu\Gamma_{\sigma\nu}^\alpha)$$

Noting that m is the mass of the test particle and $J^\alpha = eu^\alpha$, where e is the charge on the test particle, the result of the long calculation implied in (5.16) is

$$\begin{aligned}\frac{d}{ds}\left(m\frac{dz}{ds}\right)_{z=a} &= \frac{f^{3/2}S^{(1)}}{(2a)^3}\left\{\left(4\mu_1 + \frac{2m_1\mu_1}{2a}\right)\left[\frac{2m_1}{(2a)^2}\left(1+\frac{m_1}{2a}\right)+\frac{4\mu_1{}^2}{(2a)^5}\right]\right.\\ &\quad\left.-\left[\frac{4m_1\mu_1}{(2a)^2}+\frac{6\mu_1}{2a}\right]\left[\left(1+\frac{m_1}{2a}\right)^2+\frac{\mu_1{}^2}{(2a)^4}\right]\right\}\\ &\quad+\frac{f^4p^{(1)}}{(2a)^2}\left\{-2\mu_1\left[\frac{2m_1}{(2a)^2}+\frac{m_1{}^2}{4a^3}+\frac{4\mu_1{}^2}{(2a)^5}\right]\right.\\ &\quad+f^{-1}\mu_1\left[\frac{4m_1}{(2a)^3}+\frac{6m_1{}^2}{(2a)^4}+\frac{20\mu_1{}^2}{(2a)^6}\right]\\ &\quad\left.+\frac{4f^{-1}\mu_1}{2a}\left[\frac{2m}{(2a)^2}+\frac{m_1{}^2}{4a^3}+\frac{4\mu_1{}^2}{(2a)^5}\right]-\frac{6\mu_1 f^{-2}}{(2a)^2}\right\}\\ &\quad+f^{5/2}\left\{\frac{m_1e}{4a^2}\left(1+\frac{m_1}{2a}\right)^2+\frac{\mu_1{}^2e}{8a^5}\left(1+\frac{3m_1}{8a}\right)\right.\\ &\quad\left.-\tfrac{1}{2}f^{-1/2}m\left[\frac{m_1}{2a}\left(1+\frac{m_1}{2a}\right)+\frac{\mu_1{}^2}{8a^5}\right]\right\}\end{aligned} \tag{5.17}$$

There is no hope of choosing the parameters $m, e, S^{(1)}$, and $p^{(1)}$ in such a way as to make $(d/dS)(m\,dz/dS)_{z=a} = 0$. However, by expanding in powers of a^{-1} we find

$$\begin{aligned}\frac{d}{ds}\left(m\frac{dz}{ds}\right)_{z=a} &= \frac{(em_1 - mm_1)}{(2a)^2}+\frac{m_1(m_1e-mm_1)}{2(2a)^3}-\frac{29m_1{}^2(m_1e-mm_1)}{8(2a)^4}\\ &\quad-\frac{6\mu_1(p^{(1)}+S^{(1)})}{(2a)^4}+\frac{29m_1{}^3(m_1e-mm_1)}{16(2a)^5}\\ &\quad+\mu_1\left[\frac{15m_1p^{(1)}+13m_1S^{(1)}+\mu_1(4e-2m)}{(2a)^5}\right]+O(a^{-6})\end{aligned} \tag{5.18}$$

If we now force the parameters of the test particle to mimic those describing the exact solution, we must take, say,

$$S^{(1)} = -\mu_2, \qquad p^{(1)} = \mu_2, \qquad e = m = m_2$$

When these identifications are made (5.18) reduces to

$$\frac{d}{dS}\left(m_2\frac{dz}{dS}\right)_{z=a} = \frac{\mu_1}{16a^5}(m_1\mu_2 + m_2\mu_1) + O(a^{-6}) \tag{5.19}$$

This result is fairly strong evidence in favor of interpreting the line singularity between two Perjeons as a strut holding the particles in position. We should not be too worried that this is not an exact result; that is, when $(m_2\mu_1 + m_1\mu_2) = 0$ then $(d/dS)(m_2\, dz/dS)_{z=a} \neq 0$ identically. [This is easily seen from (5.17).] This is no doubt a result of replacing a Perjeon, with complicated multipole structure, by a test particle with very limited multipole structure. As a check on the calculations, (5.19) reduces to the expression given in Bonnor and Ward (1972) when $\mu_2 = 0$.

Physically, what is happening? Since the P.I.W. class has only two independent parameters, μ and m, describing four physical quantities, mass, charge, spin, and magnetic moment, it is not obvious from (5.18) what the particular interactions are. However, it is clear that the $O(a^{-2})$ term is the usual Newtonian mass-mass, charge-charge interaction and is zero only if $e = m = m_2$. If this identification is made, the elimination of certain other non-Newtonian interactions in higher orders is achieved. Equation (5.18) then becomes

$$\frac{d}{ds}\left(m_2\frac{dz}{ds}\right)_{z=a} = -\frac{6\mu_1}{(2a)^4}(p^{(1)} + S^{(1)}) + \frac{\mu_1}{(2a)^5}(15m_1p^{(1)} + 13m_1S^{(1)} + 2m_2\mu_1) + O(a^{-6}) \tag{5.20}$$

It is now fairly clear that the $O(a^{-4})$ term are the spin-spin and magnetic-dipole-magnetic-dipole interactions. The latter is classical while the former, predicted by Hawking (1972), has been discussed in detail by Wald (1972). This term will be zero only if $p^{(1)} = -S^{(1)}$ and then the $O(a^{-5})$ term in (5.19) is left. This remaining term displays an asymmetry between the source (the Perjeon) and the test particle by having μ_1 as a factor. Indeed, if $e = m$, then from (5.17) we see that if $\mu_1 = 0$, there is apparently no "force" on a test particle whereas there is a "force" if only $p^{(1)} = S^{(1)} = 0$. This behavior has also been observed by Wald and is probably a consequence of the test particle approximation. This asymmetry disallows a convincing interpretation of the $O(a^{-5})$ term. However, I still feel that the $(m_2\mu_1 + m_1\mu_2)$ factor in this term supports the view that the condition given in (2.17), of which a particular case is given in (4.4) and thus in condition (5.7), is an equilibrium condition. Indeed if a similar calculation is made using the "$e^2 = M^2$" Kerr-Newman solution as the source of the field, then again the force on a spinning test particle is found to be (test particle placed at $z = b$, with source at $z = 0$)

$$\frac{d}{ds}\left(m_1 \frac{dz}{ds}\right)_{z=b} = \frac{1}{b^2}(e_1 M_2 - m_1 M_2) + \frac{M_2{}^2}{b^3}(m_1 - e_1)$$
$$+ \frac{1}{b^4}[3a_2{}^2 M_2(m_1 - e_1) + 6M_2 a_2(S^{(1)} + p^{(1)})]$$
$$+ \frac{1}{b^5}[2a_2{}^2 M_2{}^2(e_1 - m_1) + 2M_2{}^2 a_2{}^2 e_1 - 10M_2{}^2 a_2 S^{(1)}$$
$$- 12M_2{}^2 a_2 p^{(1)}] + O(b^{-6})$$

and putting $S^{(1)} = -p^{(1)} = m_1 a_1 \qquad m_1 = e_1$ we obtain

$$\frac{d}{ds}\left(m_1 \frac{dz}{ds}\right)_{z=b} = \frac{2(M_2 a_2) m_1 M_2(a_2 + a_1)}{b^5} + O(b^{-6})$$

which gives exactly the same condition as before in order that the $O(b^{-5})$ term should be zero; that is, the sum of the angular momenta per unit mass should be zero. So again we reach the same conclusions as before as to the interpretation of condition (2.17).

6. *Conclusions*

We have attempted to show that the global integrability conditions can be plausibly interpreted as equilibrium conditions. Mathematically, of course, the integrability conditions are just smoothness conditions ensuring the regularity of the exterior geometry, when they are satisfied. The assumption then that any lack of regularity, in the exterior, in the metrics considered here, corresponds to the existence of "forces" in the system is a strong assumption, and one can imagine cases where such an assumption is inappropriate. Consider for example the NUT solution. This is generated by the monopole solution of (3.10), and one would imagine that there were no forces on this single particle. However, while the integrability condition (4.2) is satisfied for this case, (4.1) cannot be satisfied unless the contribution to Δ is taken as zero—that is, unless the NUT particle is eliminated altogether! This is perfectly consistent with the smoothness conditions since the NUT solution has a line singularity (in the exterior) stretching from the particle to $-\infty$ on the axis of symmetry. One would not say that this line singularity corresponded to the existence of a force in the system. Thus the NUT solution forces one to make stronger assumptions on the nature of particles in a system before one can connect G.I.C. with the equilibrium of the system. Much more work needs to be done to clarify this state of affairs.

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